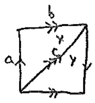
PSet 4 Solutions

1. 

over $\mathbb{Z}$: $H^* = \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z}/2$

gen H^1 is (b^*+c^*)

$(b^*+c^*)^2 = 0$

over $\mathbb{Z}/2$: $H^* = \mathbb{Z}/2 \quad \mathbb{Z}/2 \oplus \mathbb{Z}/2 \quad \mathbb{Z}/2$

gen H^1 are (b^*+c^*, a^*+c^*)

gen H^2 is $x^* \sim y^*$

$(b^*+c^*)^2 = 0$

$(a^*+c^*)^2 = y^*$

$(a^*+c^*)(b^*+c^*) = x^*$ //

2. $\Sigma X = C_1 X \cup C_2 X$, $H^i \Sigma X \cong H^i(\Sigma X, C_1 X) \cong H^i(\Sigma X, C_2 X)$

so $\alpha \sim \alpha'$, $\beta \sim \beta'$ and $\alpha' \cup \beta' = 0$ as cochains. //

3. Both integral cohom are $\mathbb{Z} \ 0 \ 0 \ \mathbb{Z}/p\mathbb{Z}$ so necessarily trivial ring struc

Now mod p cohom groups of both are $\mathbb{Z}/p \ 0 \ \mathbb{Z}/p \ \mathbb{Z}/p \ \mathbb{Z}/p$

The map $\mathbb{C}P^2 \to \mathbb{C}P^2 \cup_p D^3$, and the nontrivial cup square in $\mathbb{C}P^2$, forces $\mathbb{C}P^2 \cup_p D^3$ to have nontrivial cup square in mod p cohom.

The map $S^2 \vee S^4 \to M(\mathbb{Z}/p, 2) \vee S^4$, and the trivial cup square in $S^2 \vee S^4$, forces $M(\mathbb{Z}/p, 2) \vee S^4$ to have trivial cup square in mod p cohom. //

4. UCT $\Rightarrow$ the map on H_2 is dual to the map on H^2, but the map on H^2 is zero because the map on H^1 is zero and the gen of $H^2 T$ is a product of H^1 classes.

5. $(a \otimes b)(c \otimes d) = (-1)^{bd} ac \otimes bd = (-1)^{bd + ac + bd} ca \otimes db = (-1)^{bd + ac + bd + ad} (c \otimes d)(a \otimes b)$

$= (-1)^{(a+b)(c+d)} (c \otimes d)(a \otimes b)$ //

6. $\mathrm{Tor}(A,B) = 0$ if A or B free. Pf: if A free, use $0 \to 0 \to A \to A \to 0$ as free res; if B free then $F_1 \otimes B \hookrightarrow F_0 \otimes B$.

$\mathrm{Tor}(\mathbb{Z}/n, A) = \ker(A \xrightarrow{n} A)$ so $\mathrm{Tor}(\mathbb{Z}/n, \mathbb{Z}/m) = \mathbb{Z}/\gcd(m,n)$

Pf: $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0 \rightsquigarrow 0 \to \mathrm{Tor} \to A \xrightarrow{n} A$. //

7. $\tilde{H}^*(M(\mathbb{Z}/n, i)) = \{\mathbb{Z}/n \ \ i+1$, $\quad H^*X \otimes H^*Y \to H^*X \times Y \twoheadrightarrow \mathrm{Tor}(H^*X, H^*Y)$

$\mathbb{Z}/n \otimes \mathbb{Z}/m \quad i+j+2$

$\mathbb{Z}/n \quad i+1$

$\mathbb{Z}/m \quad j+1$

$\mathbb{Z} \quad 0$

$\mathbb{Z}/\gcd(n,m) \quad i+j+2$

$\rightsquigarrow$

$\mathbb{Z}$	$\mathbb{Z}/n$	$\mathbb{Z}/m$	$\mathbb{Z}/\gcd$	$\mathbb{Z}/\gcd$
0	$i+1$	$j+1$	$i+j+1$	$i+j+2$

(note $nm = \gcd \cdot \mathrm{lcm}$)

cf Keith Conrad, "Tensor Products", Thm 4.1.

8. $\bigoplus H_j X \otimes H_{n-j} Y \to H_n X \times Y \rightleftarrows \bigoplus \mathrm{Tor}(H_{j-1} X, H_{n-j} Y)$

$n = 2i+1$

$$\begin{array}{ccc} \mathbb{Z}/m & \xleftarrow{\cong} & \mathbb{Z}/m \\ \downarrow{\scriptstyle n/2} & \times & \downarrow \\ \mathbb{Z}/m & \longleftarrow & 0 \end{array}$$

//

9. $H_{n-1} M \cong H^1 M$, torsion would give H_0 torsion by UCT ↯ //

10.

	$k-1$	k
H^*		$\mathbb{Z}/n$
H_*	$\mathbb{Z}/n$	$\mathbb{Z}/n$

$\xleftarrow{\text{UCT}}$, $\uparrow$ PD

↯ //