Algebraic topology
Problem sheet #4
Due: 15 Jan (suggested completion of non-optional problems by 27 Nov).

1. Compute the cup product structure on the Klein bottle, with both integral and mod 2 coefficients.

2. (Optional) Prove that all cup products of positive-dimensional classes in a suspension are zero. (Cf. Hatcher 3.2.2)

3. Let X be $\mathbb{C}\mathrm{P}^2$ with an additional cell e^3 attached by a map $S^2 \to \mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^2$ of degree p, and let Y be $M(\mathbb{Z}/p, 2) \vee S^4$. Show that X and Y have isomorphic cohomology rings with integral coefficients, but not with coefficients in $\mathbb{Z}/p$. (Hatcher 3.2.8)

4. Show that every map $S^2 \to T$ induces the zero map on H_2.

5. Check that if A and B are graded commutative rings, then $A \otimes B$ is also graded commutative.

6. Compute $\mathrm{Tor}(G, H)$ for all finitely generated abelian groups G and H.

7. Compute the cohomology groups of the product of Moore spaces $M(\mathbb{Z}/n, i) \times M(\mathbb{Z}/m, j)$.

8. Show that the splitting in the homology Künneth theorem cannot be natural by considering the map $f \times \mathrm{id} : M(\mathbb{Z}/m, i) \times M(\mathbb{Z}/m, i) \to S^{i+1} \times M(\mathbb{Z}/m, i)$, where f collapses the i-skeleton of $M(\mathbb{Z}/m, i)$ to a point. (Hatcher 3.B.3)

9. Show that $H_{n-1}M$ is torsion-free for any closed orientable n-manifold M.

10. For a closed orientable $2k$-manifold M, show that if $H_{k-1}(M)$ is torsion-free then $H_k(M)$ is torsion-free. (Hatcher 3.3.25)

11. (Optional) Compute $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2k)$ as a ring. (Hatcher 3.2.5)

12. (Optional) Describe $H^*(\mathbb{C}P^\infty; \mathbb{Z})$ as a ring with finitely many multiplicative generators. (Hatcher 3.2.13)

13. (Optional) Show that there exist nonorientable 1-dimensional non-Hausdorff manifolds. (Hatcher 3.3.1)

14. (Optional) Compute the cup product structure on $H^*((S^2 \times S^8) \# (S^4 \times S^6); \mathbb{Z})$. (Hatcher 3.3.26)

15. (Optional) Show that the fundamental group of an H-space is abelian. (Hatcher 3.C.5)

16. (Optional*) Classify commutative graded Hopf algebras that are finite-dimensional in each degree, over a field of characteristic zero.

17. (Optional*) Find conditions on an H-space that ensure its cohomology is a Hopf algebra.

18. (Optional*) Find a space whose homology Pontryagin product structure is a polynomial algebra.

19. (Optional**) Compute the cohomology groups of $SO(4)$ with integral coefficients.

20. (Optional**) Compute the cohomology ring of $SO(4)$ with mod 2 coefficients.

21. (Optional***) Compute the cohomology ring of $SO(4)$ with integral coefficients.