Algebraic topology
Problem sheet #3
Due: 13 Nov.

1. Prove that if $f : S^n \to S^n$ has no fixed points, then $\deg(f) = (-1)^{n+1}$.

2. (Optional) The monoid of 2×2 matrices with integer entries, $M_2(\mathbb{Z})$, acts on $\mathbb{R}^2$ and this action descends to an action on the quotient group $\mathbb{R}^2/\mathbb{Z}^2$, which is homeomorphic to the torus T. For any $A \in M_2(\mathbb{Z})$, compute the induced map $A_* : H_2(T) \to H_2(T)$.

3. For any sequence $G_1, \ldots, G_n$ of finitely generated abelian groups, show there is a space K with $H_i(K) \cong G_i$ for $1 \leq i \leq n$ and $H_i(X) = 0$ for $i > n$.

4. (Optional) Consider the space obtained from the cube I^3 by identifying each face with the opposite face by a one-quarter right-handed-screw rotation. Compute the homology of the resulting quotient space. (Hatcher 2.2.11)

5. Show that a chain complex of free abelian groups C_n splits as a direct sum of subcomplexes of the form $0 \to L_{n+1} \to K_n \to 0$. Hint: show the short exact sequence $0 \to \operatorname{Ker} \partial \to C_n \to \operatorname{Im} \partial \to 0$ splits and take $K_n = \operatorname{Ker} \partial$. (Hatcher 2.2.43a)

6. (Optional) Let $\tilde{h}_n(X) = \prod_i \tilde{H}_i(X) / \bigoplus_i \tilde{H}_i(X)$. Show that $\tilde{h}$ satisfies the axioms for a homology theory, except that the wedge axiom fails. (Hatcher 2.3.2)

7. Directly compute the simplicial cohomology, with $\mathbb{Z}$ and $\mathbb{Z}/2$ coefficients, of $\mathbb{RP}^2$ and of the Klein bottle. (Hatcher 3.1.6b.)

8. Give an explicit singular cocycle in $C^1(S^1)$ representing a generator of $H^1(S^1)$.

9. (Optional) Show that the contravariant functors $h^n(X) := \operatorname{Hom}(H_n(X), \mathbb{Z})$ do not define a cohomology theory. (Hatcher 3.1.7)

10. Use the universal coefficient theorem to compute the cohomology of $\mathbb{RP}^3$ with coefficients in the quotient group $\mathbb{Q}/\mathbb{Z}$.

11. For all finitely generated abelian groups H, and an arbitrary abelian group G, compute $\operatorname{Ext}(H, G)$ in terms of G.

12. Let X be a space obtained from S^n by attaching an $(n+1)$-cell by a map of degree m. (This is called a Moore space and is denoted $M(\mathbb{Z}/m, n)$.) Show that the quotient $X \to X/S^n = S^{n+1}$ induces the trivial map on reduced homology, but not on cohomology. Deduce that the splitting of the universal coefficient theorem for cohomology cannot be natural. (Hatcher 3.1.11a)