Algebraic topology
Problem sheet #2
Due: 30 Oct.

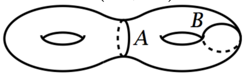
1. (Optional) Show that $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian and find a basis. (Hatcher 2.1.18)

2. Is there a short exact sequence of the form $0 \to \mathbb{Z}/4 \to \mathbb{Z}/8 \oplus \mathbb{Z}/2 \to \mathbb{Z}/4 \to 0$? For which abelian groups A is there a short exact sequence of the form $0 \to \mathbb{Z}/p^m \to A \to \mathbb{Z}/p^n \to 0$ (p prime)? Similarly for $0 \to \mathbb{Z} \to A \to \mathbb{Z}/n \to 0$? (Hatcher 2.1.14)

3. Show that chain homotopy of chain maps is an equivalence relation. (Hatcher 2.1.12)

4. Given a short exact sequence of chain complexes $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$, prove that the resulting long exact sequence $\cdots \to H_i(A) \to H_i(B) \to H_i(C) \to H_{i-1}(A) \to \cdots$ is indeed exact at position B.

5. Compute the homology of the plane with k points removed.

6. Compute $H_n(X, A)$ and $H_n(X, B)$ for X the orientable genus two surface and A and B as shown: 

(Hatcher 2.1.17b)

7. Suppose $A \subset X$ is a retract of X. Show that $H_i(X) \cong H_i(A) \oplus H_i(X, A)$.

8. (Optional) Show that $H_1(X, A)$ is not isomorphic to $\tilde{H}_1(X/A)$ when $X = [0, 1]$ and $A = 0 \cup \{\frac{1}{n}\}_{n=1}^{\infty}$. (Hatcher 2.1.26)

9. Considering the Klein bottle as a union of two Möbius strips along a circle, use the Mayer–Vietoris sequence to recompute the homology of the Klein bottle.

10. Compute the homology of the space obtained by attaching a Möbius strip to $\mathbb{RP}^2$ by a homeomorphism of its boundary circle to the standard $\mathbb{RP}^1 \subset \mathbb{RP}^2$. (Hatcher 2.2.28b)

11. The short exact sequences $0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$ always split, but this does not always yield splittings $H_n(X) \cong H_n(A) \oplus H_n(X, A)$ — why not? (Hatcher 2.2.27)

12. Prove the Brouwer fixed point theorem, that any continuous map $f : D^n \to D^n$ has a fixed point.

13. (Optional) Show that the inclusion $i : (D^n, S^{n-1}) \to (D^n, D^n - \{0\})$ is not a homotopy equivalence of pairs, i.e. there is no map of pairs $j : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ such that the composites ij and ji are homotopic to the identity through maps of pairs. (Hatcher 2.1.27b)