On the Fibrewise Poincare-Hopf Theorem

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Abstract

In this note we give a new proof of the fibrewise Poincare-Hopf theorem using Dold's euclidean-neighborhood-retract (ENR) description of the Becker-Gottlieb transfer. The fibrewise Poincare-Hopf theorem states that the transfer map for a bundle $E \to B$ with smooth manifold fibres factors through the transfer for the subbundle of zeros of a vertically-generic vertical vector field on the total space E. The ENR approach suggests certain generalizations of the theorem; we describe in particular a result expressing the transfer for the bundle $E \to B$ in terms of the transfer for the subbundle of zeros of the gradient of a fibrewise Morse-Bott function on the total space E. We illustrate the results with a discussion of the transfer for orthogonal sphere bundles over spheres.

Introduction

The classical Poincare-Hopf theorem asserts that the euler characteristic of a manifold M is the sum of the indices of the zeros of a generic vector field on M. The fibrewise Poincare-Hopf theorem correspondingly asserts that the fibrewise euler characteristic of a bundle $E \to B$ can be computed in terms of the indices of the zeros of an appropriately generic vertical vector field on the total space E.

The euler characteristic of a finite CW complex F is equal to the degree of a certain self-map of a sphere $\phi(F)\colon \mathrm{S}^n \to \mathrm{S}^n$. This map depends naturally on F, in the sense that given a fibre bundle $E \to B$, the maps $\phi(E_b)$, $b \in B$, associated to various fibres assemble to give a map $B \to \Omega^n \mathrm{S}^n$. This fibrewise euler characteristic is of course better known as the Becker-Gottlieb transfer [2]; its formulation as a fibrewise self-map of a sphere is due to Dold [5, 6].

The purpose of this note is to give a simple and transparent proof of the fibrewise Poincare-Hopf theorem for any vertically-generic vertical vector field on a smooth bundle of smooth manifolds. Though the main interest is our method of proof, this result does generalize a theorem of Brumfiel and Madsen [4, 7]; in [4] and [7] it was assumed that the bundle was associated to a principle G-bundle for a compact Lie group G and that the vector field on the total space was associated to a G-invariant vector field on the fibre. Our approach, using Dold's euclidean-neighborhood-retract (ENR) philosophy, easily extends to almost any situation in which there is a natural definition of the index of the vertical vector field. For example, we note that the fibrewise Poincare-Hopf theorem holds under the much weaker assumption that the vector field is the gradient of a fibrewise Morse-Bott function. Such a vector field is usually degenerate, and its zeros can define a subbundle of any dimension. This result can be used to decompose the transfer in terms of the indices of a series of increasingly degenerate vector fields, thereby easing computation. The ENR approach even allows us to identify the transfers of bundles with CW, not necessarily manifold, fibres in terms of subbundle transfers, but we will not discuss this case explicitly.

In section 1 we use euclidean neighborhood retractions to associate to a space F a self-map of a sphere whose degree is the euler characteristic of F. We observe that this degree is invariant under deformations of the retraction; this invariance is the key ingredient in the proofs of the main theorems. Section 2 illustrates the techniques of section 1 by proving the classical Poincare-Hopf theorem. In section 3, we use fibrewise neighborhood retractions to introduce the Becker-Gottlieb transfer as a fibrewise self-map of a sphere, and we discuss vertical deformations of the retractions. Section 4 proves the fibrewise Poincare-Hopf theorem, describes its generalization to Morse-Bott vector fields, and discusses, by way of illustration, the transfer for orthogonal sphere bundles over spheres.

This note is expository in flavor and can be considered a rearticulation and application of ideas of Dold, among others. The reader who is interested only in the statements of results should look to section 4.

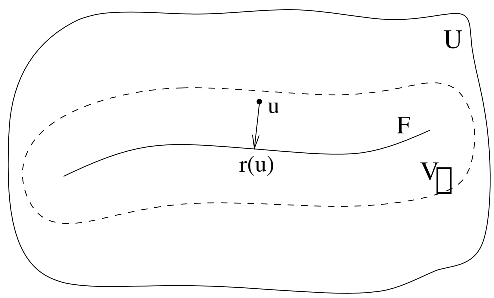
1 The Euler Characteristic

We want to associate to a space F a map $\phi(F) : \mathrm{S}^n \to \mathrm{S}^n$ for some n such that the degree of $\phi(F)$ is the euler characteristic of F. If F is a closed manifold, we can utilize the Pontryagin-Thom construction as follows. Choose an embedding of F in $\mathbb{R}^n$ and let ν_F denote the normal bundle. Notice that because F is compact, the Thom space of the normal bundle, $\mathrm{Th}(\nu_F)$, is the same as the one-point compactification of the normal bundle, $(\nu_F)^+$. Let τ_F denote the tangent bundle of F, and F_+ the union of F and a disjoint basepoint. We have the sequence of maps

$$\mathrm{S}^n \to \mathrm{Th}(\nu_F) = (\nu_F)^+ \to (\nu_F \oplus \tau_F)^+ \cong (F \times \mathbb{R}^n)^+ = F_+ \wedge \mathrm{S}^n \to S^n \qquad (1)$$

where the first map is the Pontryagin-Thom collapse map, the second is the inclusion, and the last is projection. If we identify ν_F with a tubular neighborhood U of F in $(\mathbb{R}^n)^+ = \mathrm{S}^n$, this composite map has the following form. If $p \notin U$, then p is mapped to infinity, that is to the basepoint of S^n. If $p \in U$, then p is mapped to the vector $p - \pi(p) \in \mathbb{R}^n$ where $\pi\colon U \cong \nu_F \to F \subset \mathbb{R}^n$ is the projection; that is, p is mapped to the vector to which it corresponds in the embedded normal bundle.

This construction did not depend essentially on F being a manifold—it required only the existence of an embedding of F into some $\mathbb{R}^n$ with a neighborhood U of the image and a retraction $U \to F$. Dold called such spaces euclidean neighborhood retracts (ENRs) and studied them and their fibrewise analogs in [5] and [6]. For now we content ourselves with the observation that any finite CW complex F is an ENR and we imitate the construction of sequence 1 in this context. Choose an embedding of F in some $\mathbb{R}^n$, a neighborhood U of F, and a retraction $r\colon U \to F$. This neighborhood U may be quite badly behaved, so we consider a smaller, more uniform neighborhood which resembles a tubular neighborhood in the case when F is a manifold. This smaller neighborhood is $V_\epsilon = \{u \in U \text{ s.t. } |u - r(u)| < \epsilon\}$. The picture is as follows:



On the one hand, V_ϵ is openly embedded in $\mathbb{R}^n$, and on the other hand it maps properly to D^n_ϵ, the n-disc of radius ϵ:

$$\begin{array}{ccccc} \mathbb{R}^n & \xleftarrow{\text{open}} & V_\epsilon & \xrightarrow{\text{proper}} & \mathrm{D}^n_\epsilon \cong \mathbb{R}^n \\ u & \longleftarrow\!\shortmid & u & \longmapsto & u - r(u) \end{array}$$

Recall that one-point compactification is contravariant with respect to open embeddings and covariant with respect to proper maps, so one-point compactifying this sequence gives a map $\phi(F)\colon S^n \to S^n$. We will also sometimes denote this map $\phi(F, r)$ when we want to emphasize its dependence on the retraction.

Definition 1.1. The euler characteristic of F is the degree of $\phi(F)$, that is $\chi(F) = \deg \phi(F)$.

Implicit in this definition is the fact that the degree does not depend on the choice of embedding, neighborhood, retraction, or ϵ. This is easily checked as follows. Fixing the other choices, varying ϵ certainly does not affect the degree. Now for a fixed embedding with two different neighborhood retractions (U, r) and (U', r'), use the fact that the embedding is a cofibration to produce a retraction $\tilde{r}$ of a neighborhood $\tilde{U}$ of $F \times I$ in $\mathbb{R}^n \times I$, agreeing with r and r' in some neighborhoods of $F \times \{0\}$ and $F \times \{1\}$ respectively. Performing the above one-point compactification construction on $(F \times$

$I, \tilde{U}, \tilde{r})$ produces a homotopy between $\phi(F, r)$ and $\phi(F, r')$. Similarly, if we increase n sufficiently (which does not affect the degree), any two embeddings are isotopic, and the one-point compactification construction on the isotopy produces a homotopy as desired. In section 2 we will further justify this definition by showing that it agrees with more classical definitions of the euler characteristic.

We now make two key observations about the construction of $\phi(F, r)$. First, the remarks in the above paragraph show in particular that any two retractions $r, r' : U \to F$ are homotopic when restricted to some neighborhood of F. In fact, given *any* map $s : U \to F$ homotopic to a retraction $r : U \to F$, the construction, exactly as given, produces a map $\phi(F, s) : \mathrm{S}^n \to \mathrm{S}^n$, and the degree of $\phi(F, s)$ is equal to the degree of $\phi(F, r)$, as is easily verified. Second, as removing the origin $\{0\}$ from $(\mathbb{R}^n)^+ = \mathrm{S}^n$ results in a contractible space, the degree of $\phi(F, s)$ depends only on a neighborhood of the inverse image of $\{0\}$, that is on a neighborhood of the set of fixed points of s, $\mathrm{Fix}(s) = \{u \in U \text{ s.t. } u - s(u) = 0\}$. When s is a retraction this fixed set is precisely F, but in general it may be only a subset of F.

We will utilize these observations primarily as follows. Let $f : F \to F$ be a map homotopic to the identity on F such that the fixed point set of f, $\mathrm{Fix}(f)$, is a discrete subset of F. Then for any retraction $r : U \to F$, the composite $fr : U \to F$ is homotopic to r and so the degree of $\phi(F, fr)$ is the euler characteristic of F. But this degree depends only on the behavior of fr in a neighborhood of $\mathrm{Fix}(fr) = \mathrm{Fix}(f)$ in U, which is in turn determined up to homotopy by the behavior of f on a neighborhood of $\mathrm{Fix}(f)$ in F.

2 The Poincare-Hopf Theorem

The euler characteristic is most simply defined for a finite simplicial complex as the alternating sum of the number of simplices in each dimension. For a manifold, the euler characteristic may instead be given as the sum of the indices of the vanishing points of a generic vector field. The (classical) Poincare-Hopf theorem asserts that whenever both are defined, these quantities agree. We prove this by relating each to $\chi(F) = \deg \phi(F)$ as defined in the last section.

We begin by recalling the definition of the index of a vanishing point of a vector field. Let F be a closed manifold and X a generic vector field. (Generic here means transverse to the zero section as a map $F \to \mathrm{T}F$.) Now if $X(p) = 0 \in \mathrm{T}_pF$ then the derivative map is $\mathrm{d}X_p : \mathrm{T}_pF \to \mathrm{T}_{(p,0)}\mathrm{T}F \cong \mathrm{T}_pF \oplus \mathrm{T}_pF$. Composing with one projection to T_pF gives the identity map, and composing with the other gives a map $\mathrm{T}_pF \to \mathrm{T}_pF$ that we will also denote by $\mathrm{d}X_p$.

Definition 2.1. The index of a generic vector field X on the manifold F, at a point p where X vanishes, is the degree of the one-point compactification of the derivative

$\mathrm{d}X_p \colon \mathrm{T}_pF \to \mathrm{T}_pF$. That is, $\mathrm{ind}_X(p) = \deg(\mathrm{d}X_p^+)$.

Proposition 2.2. *Let X be a generic vector field on a closed manifold F and let $Z(X)$ denote the set of points where X vanishes. Then $\chi(F) = \sum\limits_{p \in Z(X)} \mathrm{ind}_X(p)$ where $\chi(F)$ is the degree given in definition 1.1.*

Proof. Integrate $-X$, the negative of the vector field X, to a self-map $f \colon F \to F$ and take $\chi(F)$ to be $\deg \phi(F, fr)$ as defined in section 1. Note that we integrate $-X$, not X, for the simple reason that $\phi(F, fr)$ is locally $\mathrm{id}_U - fr$, so a deformation f along $-X$ deforms ϕ in the direction of X. The vanishing set or zeros $Z = Z(X)$ of the vector field are precisely the fixed points of the self-map f. Further, it is clear that the index of X at a point p is the same as the local degree of $\phi(F, fr)$ at p; (by the local degree we mean the degree of the restriction of $\phi(F, fr)$ to a small neighborhood of p).

This equivalence between the index of X and the local degree of $\phi(F, fr)$ may be seen in more detail as follows. Choose the embedding of F in $\mathbb{R}^n$ so that the image of a neighborhood of Z is flat in $\mathbb{R}^n$; that is, the neighborhood is mapped diffeomorphically onto a collection of open subsets of affine linear subspaces of $\mathbb{R}^n$. Let the retraction $r \colon U \to F$ of a neighborhood U of F be given by orthogonal projection to F in a small neighborhood of Z. Now deform the vector field X so that it is linear in a similarly small neighborhood of Z; that is, in this neighborhood, the deformed vector field X agrees with its derivative $\mathrm{d}X|_Z$, using the canonical identification of a tangent space to $\mathbb{R}^n$ with $\mathbb{R}^n$. Now take $f \colon F \to F$ to be a unit-time integral of $-X$, and more generally $f_t \colon F \to F$ to be a time-t integral. Consider the homotopy

$$U \times I \xrightarrow{h} \mathbb{R}^n$$
$$(u, t) \mapsto \begin{cases} (u - ru) + \frac{(ru - f_t ru)}{t} & \text{if } t > 0 \\ (u - ru) + X(ru) & \text{if } t = 0 \end{cases}$$

between $u \mapsto u - fru$ and $u \mapsto u - ru + X(ru)$. Let $\widetilde{V}_\epsilon$ be the neighborhood of $Z \times I$ given by $\{(u, t) \in U \times I \text{ s.t. } |(u - ru) + \frac{(ru - f_t ru)}{t}| < \epsilon\}$. Then for very small ϵ, the map $h|_{\widetilde{V}_\epsilon} \colon \widetilde{V}_\epsilon \to \mathrm{D}^n_\epsilon$ is proper. One-point compactification yields the desired homotopy between $\phi(F, fr)$ and a suspension of the index map.

Roughly, this is just to say that locally at a zero, the derivative $\mathrm{d}X$ is homotopic to the vector field X, which is in turn homotopic to its integration $-f$. We belabor this point because it will arise in a fibrewise setting in section 4. $\square$

Next we compare $\chi(F)$ to the classical alternating sum definition of the euler characteristic.

Proposition 2.3. *Let F be a finite simplicial complex and let $|F_i|$ denote the order of the set of i-simplices of F. Then $\chi(F) = \sum\limits_i (-1)^i |F_i|$.*

Proof. By induction we construct a self-map $f\colon F \to F$ with fixed point set $\mathrm{Fix}(f)$ equal to the set of barycenters of simplices of F. The local degree of $\phi(F, fr)$ at the barycenter of a simplex of dimension i will be $(-1)^i$. (As in the proof of Proposition 2.2, the local degree of $\phi(F, fr)$ at a point p refers to the degree of the restriction of $\phi(F, fr)$ to the component of V_ϵ containing p, for ϵ sufficiently small.)

For each k-simplex σ, fix a homeomorphism $h\colon \sigma \to \mathrm{D}^k$ of σ with the unit disc in $\mathbb{R}^k$ such that h takes the barycenter of σ to the origin. Let f be the identity on the 0-simplices of F and suppose f has been defined on the $(k-1)$-skeleton of F. Define f on a k-simplex σ to be the composite

$$\sigma \xrightarrow{h} \mathrm{D}^k \xrightarrow{c} \mathrm{D}^k \xrightarrow{e} \mathrm{D}^k \xrightarrow{h^{-1}} \sigma,$$

where $c\colon \mathrm{D}^k \to \mathrm{D}^k$ is the cone on $hfh^{-1}\colon S^{k-1} \to S^{k-1}$, and $e\colon \mathrm{D}^k \to \mathrm{D}^k$ is a radially expanding map such as $t \mapsto (2 - |t|)t$.

The resulting $f : F \to F$ maps a point in the interior of a simplex σ to a point further away from the barycenter of σ. In a neighborhood of the barycenter of σ, the euler characteristic map $\phi(F, fr)$, given by $\mathrm{id}_U - fr$, will therefore be a reflection in i hyperplanes, where i is the dimension of σ. Thus the deformation f contributes the appropriate local degree at the fixed points. $\square$

3 The Becker-Gottlieb Transfer

Let $\eta\colon E \to B$ be a fibre bundle, $b \in B$ a point in the base, and E_b the fibre over b. If we can construct the euler characteristic map $\phi(E_b)\colon \mathrm{S}^n \to \mathrm{S}^n$ of section 1 in a way that depends continuously on b, then we will have produced a map $B \to \Omega^n\mathrm{S}^n$. This map will be well defined up to homotopy after composition with $\Omega^n\mathrm{S}^n \to \mathrm{Q}(S^0)$. Recall the sequence 1 in section 1, in which the map $\phi(E_b)$ factored as $\mathrm{S}^n \to \mathrm{S}^n \wedge (E_b)_+ \to \mathrm{S}^n$. Instead of projecting to S^n we can compose the first map with the inclusion $\mathrm{S}^n \wedge (E_b)_+ \to \mathrm{S}^n \wedge E_+$. This produces a refined fibrewise euler characteristic $B \to \Omega^n(\mathrm{S}^n \wedge E_+) \to \mathrm{Q}(E_+)$ known as the Becker-Gottlieb transfer for the bundle $E \to B$.

We suppose for simplicity that B is compact and that the fibre of $\eta\colon E \to B$ is a finite CW complex. These conditions can be relaxed to, for example, B paracompact and E a euclidean neighborhood retract over B; see [5]. We now perform the construction of section 1 simultaneously on all fibres of η. Choose an embedding $E \to B \times \mathbb{R}^n$ and a neighborhood U of E with a retraction $r : U \to E$ commuting with projection to B. Replace U by a smaller uniform neighborhood $V_\epsilon = \{u \in U \text{ s.t. } |\pi(u) - \pi(r(u))| < \epsilon\}$ where $\pi\colon B \times \mathbb{R}^n \to \mathbb{R}^n$ is projection. This gives the sequence of maps

$$\begin{array}{ccccc} B \times \mathbb{R}^n & \xleftarrow{\text{open}} & V_\epsilon & \xrightarrow{\text{proper}} & E \times \mathrm{D}^n_\epsilon \cong E \times \mathbb{R}^n \\ u & \longleftarrow & u & \longmapsto & (r(u), \pi(u) - \pi(r(u))) \end{array}$$

One-point compactification yields a map $\chi(\eta) = \chi(\eta, r): B_+ \wedge \mathrm{S}^n \to E_+ \wedge \mathrm{S}^n$, whose adjoint we may compose with the inclusion $\Omega^n(\mathrm{S}^n \wedge E_+) \to \mathrm{Q}(E_+)$. The composite depends on the choices only up to homotopy.

Definition 3.1. The Becker-Gottlieb transfer for the bundle $\eta : E \to B$ is the map $\chi(\eta)\colon B \to \mathrm{Q}(E_+)$ constructed above.

The Becker-Gottlieb transfer is of course due to Becker and Gottlieb, and has had many formulations since its introduction; in particular this version harkens back to Dold, though we learned it from Segal.

The key observation, as in section 1, is that the construction does not depend on $r : U \to E$ being a retraction, so long as it remains a map over B. In particular, let $f\colon E \to E$ be a vertical deformation of the identity on E — that is, f is homotopic to the identity and commutes with projection to B. In this case, the transfers constructed using $r\colon U \to E$ and $fr\colon U \to E$ are homotopic, that is $\chi(\eta, r) \sim \chi(\eta, fr)\colon B \to \mathrm{Q}(E_+)$. Further, as before, the homotopy class of $\chi(\eta, fr)$ depends only on the behavior of $fr\colon U \to E$ in a neighborhood of the fixed points of fr, which in turn is determined by the behavior of $f\colon E \to E$ in a neighborhood of the fixed points of f.

4 The Fibrewise Poincare-Hopf Theorem

We now relate the Becker-Gottlieb transfer of a fibre bundle $\eta\colon E \to B$ to the fibrewise index of an appropriately generic vertical vector field on E.

Suppose the fibre F of η is a closed (smooth) manifold, and let X be a vertical vector field on E that restricts to a generic vector field on every fibre. Note that in general there are obstructions to the existence of such a vector field. The zero set $Z = Z(X)$ of such an X is a covering space of B, $Z \to B$, and we define the fibrewise index of X along Z. Let $\mathrm{T}E$ denote the (vertical) tangent bundle of E. As we saw in section 2, the tangent bundle $\mathrm{T}(\mathrm{T}E)$ of this bundle, when restricted to the zero section of $\mathrm{T}E$, splits as two copies of $\mathrm{T}E$. Thus the derivative of X along Z is $\mathrm{d}X|_Z\colon \mathrm{T}E|_Z \to \mathrm{T}(\mathrm{T}E)|_{\tilde{Z}} \cong \mathrm{T}E|_Z \oplus \mathrm{T}E|_Z$ where $\tilde{Z}$ is the zero section of $\mathrm{T}E$ over Z. Composing with one projection gives the identity, and composing with the other gives a map that we also refer to as the derivative $\mathrm{d}X\colon \mathrm{T}E|_Z \to \mathrm{T}E|_Z$.

Definition 4.1. Let X be a vertically generic vector field on the total space of a bundle $E \to B$ as above, with zero set Z. Choose a bundle ν over Z complementary to $\mathrm{T}E|_Z$. The one-point compactification of the map

$$Z \times \mathbb{R}^n \cong \mathrm{T}E|_Z \oplus \nu \xrightarrow{\mathrm{d}X \oplus \mathrm{id}} \mathrm{T}E|_Z \oplus \nu \cong Z \times \mathbb{R}^n$$

is called the index of X, denoted $\mathrm{Ind}(X) : Z_+ \wedge S^n \to Z_+ \wedge S^n$. We also refer to the stabilized n-fold loop of this map as the index, $\mathrm{Ind}(X)\colon \mathrm{Q}(Z_+) \to \mathrm{Q}(Z_+)$.

Theorem 4.2. *Let $\eta\colon E \to B$ be a fibre bundle over a compact base B with fibre F a closed manifold. Let X be a vertical vector field on E restricting to a generic vector field on each fibre. Denote the zero set of X by Z and the projection by $\eta_S : Z \to B$. The diagram*

$$\begin{tikzcd}
B \arrow[r, "\chi(\eta)"] \arrow[d, "\chi(\eta_S)"'] & \mathrm{Q}(E_+) \\
\mathrm{Q}(Z_+) \arrow[r, "\mathrm{Ind}(X)"'] & \mathrm{Q}(Z_+) \arrow[u, "\mathrm{Q}(\mathrm{inc})"']
\end{tikzcd} \tag{2}$$

commutes up to homotopy. Here $\chi(\eta)$ and $\chi(\eta_S)$ are the transfers for the corresponding bundles, $\mathrm{Ind}(X)$ *is the index, and* $\mathrm{Q}(\mathrm{inc})$ *is the inclusion.*

As mentioned in the introduction, this is a slight generalization of the theorem of Brumfiel and Madsen [4] as presented in [7]. There it is assumed that η is associated to a principle G-bundle, for a compact Lie group G, namely $\eta\colon P \times_G F \to B$, and that the vector field X is associated to a generic G-invariant vector field on F.

Proof. As in the proof of Proposition 2.2, we integrate $-X$ to a map $f : E \to E$ and use $\chi(\eta, fr)$ as our model of the transfer $\chi(\eta)$, where r is any retraction. Already with this substitution it is clear that the diagram 2 will commute up to homotopy, but we can make the homotopy more explicit by constructing intermediate maps as shown:

$$\begin{tikzcd}
B \arrow[r, "\chi(\eta,fr)"] \arrow[d, "\chi(\eta_S)"'] \arrow[dr, bend left=10, "\widetilde{\chi(\eta,fr)}"'] & \mathrm{Q}(E_+) \\
\mathrm{Q}(Z_+) \arrow[r, bend left, "\mathrm{Ind}(X,fr)"] \arrow[r, bend right, "\mathrm{Ind}(X)"'] & \mathrm{Q}(Z_+) \arrow[u, "\mathrm{Q}(\mathrm{inc})"']
\end{tikzcd} \tag{3}$$

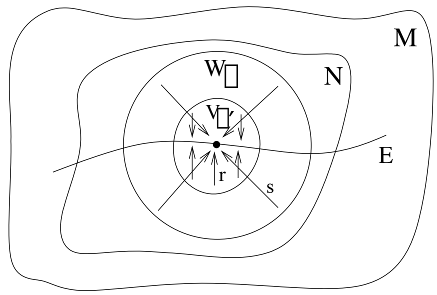
In order to construct these intermediate maps, we need to make various choices of embeddings and neighborhoods for the bundles η and η_S. By making compatible choices, we ease the proof of commutativity. Embed E in $B \times \mathbb{R}^n$ and let M be a neighborhood of E with retraction $r\colon M \to E$. The subset Z of E is thereby embedded in $B \times \mathbb{R}^n$. Choose a neighborhood N of Z such that $N \subset M$, and a retraction $s\colon N \to Z$. The diagram is

$$\begin{tikzcd}
E \arrow[r, hook, shift left] & M \arrow[l, shift left, "r"] \arrow[r, hook] & B \times \mathbb{R}^n \\
Z \arrow[u, hook] \arrow[r, hook, shift left] & N \arrow[l, shift left, "s"] \arrow[u, hook] &
\end{tikzcd}$$

Next take uniform subneighborhoods

$$
\begin{aligned}
W_\epsilon &= \{u \in N \text{ s.t. } |\pi(u) - \pi(s(u))| < \epsilon\} \\
V_{\epsilon'} &= \{u \in M \text{ s.t. } |\pi(u) - \pi(fr(u))| < \epsilon'\}
\end{aligned}
$$

where $\epsilon' \ll \epsilon$. The picture is as follows:



These choices yield the diagram of maps

$$
\begin{tikzcd}
B \times \mathbb{R}^n & V_{\epsilon'} \arrow[l, "o"'] \arrow[r, "p"] \arrow[dl, "o"] \arrow[ddl, "o"] \arrow[ddr, "p"] & E \times \mathrm{D}^n_{\epsilon'} \\
W_\epsilon \arrow[u, "o"'] \arrow[d, "p", "\cong"'] & & \\
Z \times \mathrm{D}^n_\epsilon & & Z \times \mathrm{D}^n_{\epsilon'} \arrow[uu] \arrow[uul, phantom, "h"]
\end{tikzcd}
\tag{4}
$$

given by

$$
\begin{tikzcd}
u & u \arrow[l, mapsto] \arrow[r, mapsto] \arrow[dl, mapsto] \arrow[ddl, mapsto] \arrow[ddr, mapsto] & (r(u), \pi(u) - \pi(fr(u))) & v \\
u \arrow[u, mapsto] \arrow[d, mapsto] & & & \\
(s(u), \pi(u) - \pi(s(u))) & & (s(u), \pi(u) - \pi(fr(u))) & v \arrow[uu, mapsto]
\end{tikzcd}
$$

Here the maps labeled "o" and "p" are open and proper respectively, the indicated triangles are commutative or homotopy commutative, and the lower left map is a homeomorphism. Notice that one-point compactifying the left two maps, taking the adjoint, and then looping gives $\chi(\eta_S)$, similarly for the top maps gives $\chi(\eta, fr)$, and one-point compactifying and looping the right map gives Q(inc).

Now using this diagram, define $\widetilde{\chi(\eta, fr)}$ to be the looped adjointed one-point compactification of the sequence $B \times \mathbb{R}^n \leftarrow V_{\epsilon'} \rightarrow Z \times \mathrm{D}^n_{\epsilon'}$. This is a lift up to homotopy of $\chi(\eta, fr)$ along the inclusion $Z \times \mathrm{D}^n_{\epsilon'} \rightarrow E \times \mathrm{D}^n_{\epsilon'}$. Similarly, define $\mathrm{Ind}(X, fr)$ to be the looped one-point compactification of the sequence $Z \times \mathrm{D}^n_{\epsilon} \leftarrow V_{\epsilon'} \rightarrow Z \times \mathrm{D}^n_{\epsilon'}$. This map is, so to speak, an integration of $\mathrm{Ind}(X)$, in that it enacts on a neighborhood of Z what $\mathrm{d}X$ does on an infinitesimal neighborhood of Z, that is on the tangent bundle at Z.

The (homotopy) commutativity of diagram 4 immediately implies the homotopy commutativity of the two triangles in diagram 3. It remains only to see that $\mathrm{Ind}(X, fr)$ is homotopic to $\mathrm{Ind}(X)$. This can be achieved as in the proof of Proposition 2.2. That is, choose an embedding of E in $B \times \mathbb{R}^n$ that is flat in a neighborhood of Z and presume the retraction is given by orthogonal projection near this region. Deform X to be linear near Z, and then construct a homotopy, using shorter and shorter integrations f_t of $-X$, from $u \mapsto \pi(u) - \pi(fru)$ to $u \mapsto \pi(u) - \pi(ru) + \pi(X(ru))$. One-point compactification on a small neighborhood of Z yields the desired homotopy from $\mathrm{Ind}(X, fr)$ to $\mathrm{Ind}(X)$. $\square$

This result does not depend on the zero set $Z(X)$ being fibrewise discrete. Indeed, we need only have that Z is a fibre bundle over B and that X is well enough behaved that we can define a sensible index map. One such situation is when X is the gradient of a fibrewise Morse-Bott function ψ on E, that is a function ψ restricting to a Morse-Bott function on each fibre. Recall that a Morse-Bott function $\psi : F \rightarrow \mathbb{R}$ is one whose critical set is a submanifold S of F such that the Hessian of ψ is nondegenerate on the normal bundle to S in F. Given such an X, the derivative $\mathrm{d}X$ gives an automorphism of the normal bundle ν_Z of Z in E. Choose a complementary bundle μ such that the sum $\nu_Z \oplus \mu$ has constant dimension, that is $\nu_Z \oplus \mu \cong Z \times \mathbb{R}^n$ for some n. One-point compactification of $\mathrm{d}X \oplus \mathrm{id}$ yields the index map $\mathrm{Ind}(X)$ as before.

Theorem 4.3. *Let $\eta\colon E \rightarrow B$ be a fibre bundle over a compact base with closed manifold fibre. Let X be a vertical vector field on E that is the gradient of a fibrewise Morse-Bott function. Denote the zero set of X by Z, the bundle $Z \rightarrow B$ by η_S, the index of X by* $\mathrm{Ind}(X)$*, and the transfers for the two bundles $\chi(\eta)$ and $\chi(\eta_S)$ respectively. The diagram*

$$\begin{tikzcd} B \arrow[r, "\chi(\eta)"] \arrow[d, "\chi(\eta_S)"'] & \mathrm{Q}(E_+) \\ \mathrm{Q}(Z_+) \arrow[r, "\mathrm{Ind}(X)"'] & \mathrm{Q}(Z_+) \arrow[u, "\mathrm{Q}(\mathrm{inc})"'] \end{tikzcd}$$

commutes up to homotopy.

The proof is the same as for the previous theorem. This theorem 4.3 can be applied iteratively, computing the transfer of a bundle in terms of a subbundle, then the transfer of the subbundle in terms of its subbundles, and so on.

$\sim\sim\sim$

We conclude with a simple transfer computation which utilizes theorems 4.2 and 4.3. Let $\Phi : \mathrm{S}^{n-1} \to \mathrm{SO}(k+1)$ be the clutching map defining an orthogonal S^k-bundle over S^n, with $n > 2$. This S^k-bundle ϕ is the unit sphere bundle of the vector bundle φ associated to the clutching map Φ. Let $\mathrm{J}(\Phi) \in \pi^{\mathrm{s}}_{n-1}$ denote the image of the map Φ under the J-homomorphism. We compute the fibrewise euler characteristic $\mathrm{S}^n \xrightarrow{\chi(\phi)} \mathrm{Q}(E_+) \to \mathrm{Q}(\mathrm{S}^0)$ of the bundle ϕ in terms of $\mathrm{J}(\Phi)$.

Using a Morse-Bott function on the fibrewise suspension $\Sigma\phi$ of ϕ we express the transfer of the bundle ϕ in terms of the transfer of its suspension; then using a (different) Morse function we compute the transfer of this suspension. Consider $\Sigma\phi$ to be the unit sphere bundle in the direct sum $\varphi \oplus \underline{\mathbb{R}}$ of the $\mathbb{R}^{k+1}$-bundle φ with the trivial line bundle $\underline{\mathbb{R}}$ over S^n and let $h : \varphi \oplus \underline{\mathbb{R}} \to \underline{\mathbb{R}} \to \mathbb{R}$ denote projection. The function h^2 is fibrewise Morse-Bott on $\Sigma\phi$ and its gradient has zero set $\phi \sqcup \mathrm{S}^n \sqcup \mathrm{S}^n$, that is the sphere bundle ϕ in $\varphi \times \{0\}$ disjoint union the two trivial bundles $\mathrm{S}^n \to \mathrm{S}^n$ in $\varphi \times \{1\}$ and $\varphi \times \{-1\}$. The index map is the identity at the two S^n components and is a reflection along $\underline{\mathbb{R}}$ at ϕ. Let $\overline{\chi}(\eta)$ denote the transfer $\chi(\eta) : B \to \mathrm{Q}(E_+)$ of a bundle η composed with the map $\mathrm{Q}(E_+) \to \mathrm{Q}(\mathrm{S}^0)$. Theorem 4.3 implies that $\overline{\chi}(\Sigma\phi) = 2 - \overline{\chi}(\phi)$.

Alternatively, consider the Morse function h on $\Sigma\phi$. Its gradient has zero set $\mathrm{S}^n \sqcup \mathrm{S}^n$. The index is trivial on one S^n and thus (by 4.2) the transfer is $\overline{\chi}(\Sigma\phi) = 1 + \tau$, where $\tau : \mathrm{S}^n \to \mathrm{Q}(\mathrm{S}^0)$ is given as follows. Choose a complementary vector bundle ω to the bundle φ and identify $\varphi \oplus \omega$ with the trivial bundle $\mathbb{R}^N \times \mathrm{S}^n \to \mathrm{S}^n$. Define a map Θ as follows:

$$\begin{aligned} &\mathrm{S}^n \xrightarrow{\Theta} \mathrm{O}(N) \\ &x \mapsto (\mathbb{R}^N \cong \varphi_x \oplus \omega_x \xrightarrow{-1 \oplus 1} \varphi_x \oplus \omega_x \cong \mathbb{R}^N). \end{aligned}$$

The transfer summand τ is the stabilized one-point compactification of Θ. Notice that $\tau = \mathrm{J}(\Theta) + \deg(\tau)$, where $\deg(\tau) \in \{\pm 1\}$ is the image component of τ in $\mathrm{Q}(\mathrm{S}^0)$; this degree $\deg(\tau)$ is plus or minus one as the dimension of the fibre sphere S^k is odd or even respectively. Given the earlier calculation of $\overline{\chi}(\Sigma\phi)$ in terms of $\overline{\chi}(\phi)$, we have

$$\overline{\chi}(\phi) = 1 - \tau = 1 - \deg(\tau) - \mathrm{J}(\Theta) = \chi(\mathrm{S}^k) - \mathrm{J}(\Theta).$$

We identify Θ as (stably) $(-1)^{k+1}(\eta \cdot \Phi)$, where η is now the nontrivial element of the first stable stem π^{s}_1. Let ϑ denote the vector bundle on S^{n+1} corresponding to the clutching map Θ, as φ is the bundle on S^n corresponding to Φ. In the chain complex formulation of k-theory ([1], section 10), the bundle η on $(\mathrm{S}^1, *) \cong ([-1,1], \{-1,1\})$ is represented by the following complex of bundles on $\mathrm{I} = [-1,1]$:

$$0 \to \underline{\mathbb{R}} \xrightarrow{t} \underline{\mathbb{R}} \to 0.$$

Here the map is multiplication by $t \in \mathrm{I}$ on the fibre $\underline{\mathbb{R}}_t$. The product bundle $\eta \otimes \varphi$ on $(\mathrm{S}^{n+1}, *) \cong (\mathrm{I} \times \mathrm{S}^n, (\{-1,1\} \times \mathrm{S}^n) \cup (\mathrm{I} \times *))$ is then given by the complex

$$0 \to \underline{\mathbb{R}} \otimes \varphi \xrightarrow{t \otimes 1} \underline{\mathbb{R}} \otimes \varphi \to 0.$$

This complex of bundles on $\mathrm{I} \times \mathrm{S}^n$ is simply

$$0 \to \varphi \xrightarrow{t} \varphi \to 0,$$

where we have not distinguished between the bundle φ on S^n and its pullback to $\mathrm{I} \times \mathrm{S}^n$; similarly we let ω also denote the pullback of ω to $\mathrm{I} \times \mathrm{S}^n$. Adding an acyclic complex to the preceding complex gives

$$0 \to \varphi \oplus \omega \xrightarrow{t \oplus 1} \varphi \oplus \omega \to 0.$$

The k-theory class of this complex on $(\mathrm{I} \times \mathrm{S}^n, (\{-1,1\} \times \mathrm{S}^n) \cup (\mathrm{I} \times *))$ corresponds to the reduced k-theory class of ϑ on S^{n+1}. This reduced identification $\vartheta \cong \eta \otimes \varphi$ gives a homotopy $\Theta \simeq (-1)^{k+1}(\eta \cdot \Phi)$ of maps from S^n to the stable orthogonal group O; (note that the sign merely ensures that the maps land in the same component of O).

This description of Θ implies that

$$\overline{\chi}(\phi) = \chi(\mathrm{S}^k) - \mathrm{J}(\Theta) = \chi(\mathrm{S}^k) + \eta \cdot \mathrm{J}(\Phi) \in [\mathrm{S}^n, \mathrm{Q}(\mathrm{S}^0)],$$

where we have used the fact that the J-homomorphism commutes with multiplication by η in degrees larger than 2, and that η has order 2. This result agrees with the homotopy-theoretic computation of the transfer for sphere bundles in [3].

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